FRSF Award Proposal

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On Isoperimetric Inequalities in Euclidean/Minkowski Spaces

1. Results from prior FRSF supports

The proposer was awarded the FRSF Award, #911, in the amount of $\$3,300$, for AY 2006/2007, for the project "Convex Figures of Constant Width in Euclidean/Minkowski Spaces" to visit Chemnitz University of Technology, Germany for two weeks in the summer of 2007. The purpose of this visit was to work jointly with Professor Horst Martini on the project. The first part of the project "On Reuleaux Triangles in Minkowski Planes" was published in the "Beiträge zur Algebra und Geometrie/Contributions to Algebra and Geometry" Journal, **48** (2007), 225-235. The second part of the project "A Construction of Convex Figures of Constant Width" was published in the "Computer Aided Geometric Design" Journal, **25** (2008), 751-755. This paper has also caught the attention of the Wolfram Research, Inc. Following is the e-mail from the Wolfram Research representative:

"Your article, "A construction of convex figures of constant width", caught the attention of one of my colleagues, who thought it could be developed into an interesting Demonstration to add to the Wolfram Demonstration Project. The Demonstration Project, launched alongside Mathematica 6 in May 2007, is a collection of over 2000 interactive Demonstrations that cover myriad, interests, and skill levels. Your published Demonstration will appear on the Wolfram Demonstration website, which averages over 50,000 hits a week."

The proposer was awarded the FRSF Award, #945, in the amount of $\$2,700$, for AY 2007/2008, for the project "Properties of Convex Figures of Constant Width in Euclidean/Minkowski Spaces" to visit Chemnitz University of Technology, Germany for two weeks in the summer of 2008. The purpose of this visit was to continue working on the project with Professor Horst Martini. The joint paper based on the project has been accepted for publication in the "Rocky Mountain Journal of Mathematics."

2. Background and Definitions

It is well-known that *among all planar regions with a given perimeter, the circle enclosed the greatest area.* This statement is most succinctly expressed in the *isoperimetric inequality:*

$$L^2 \geq 4\pi A,$$

where L is the length of the boundary curve and A is the area of the region. The equality sign holds if and only if the region is a circle.

Maximizing an area by a given length is a problem that dates back to ancient time. References to this theorem appear in literary history as far back as 100 BC and can be found in Virgil's Aeneid and the saga of Queen Dido. In this account, the Queen demonstrates a mathematical ability in addition to possessing keen entrepreneurial skills which are to her advantage when having to overcome severed tragedies. One of these involved fleeing her native city of Tyria in ancient Phoenicia.

She arrived in North Africa, and approached a local chief. She proposed that she be given as much land as she could isolate with the skin of an ox. The local chief jumped at the opportunity thinking this offer was too good to pass up. It was agreed to, and a large ox was sacrificed for its hide. Queen Dido sliced the hide into small strips and lay these out along a semicircle, using the North African coast as the supplementary boundary. She obtained in this way the maximal area she could get by such a procedure. Evidently, Queen Dido knew the isoperimetric inequality, and understood how to use this fact to find the best solution to her problem, which uses a semicircle rather than a circle.

In the Eucliden space, the isoperimetric theorem has generalizations to higher dimensions, and even has many variants in the Euclidean plane. For example, one version states that among all polygons with k sides and a fixed perimeter, those that are perfectly symmetric (i.e., regular) have the greatest area. These area and volume optimization theorems are especially appealing because they offer physical insights into nature. For example, they tell us that in order to reduce the amount of surface that is exposed to the cold on a winter night, a cat will curl up. They help us understand why honeybees build hives with cells that are perfectly hexagonal in shape. The isoperimetric theorem also helps explain why water pipes should have a round cross-section.

The isoperimetric inequality can also be extended to surfaces in the three-dimensional Euclidean space. Namely, among all simple closed surfaces with given surface area, the sphere encloses a region of maximum volume. An analogous result holds in Euclidean space of any dimension.

Although most of the work on isoperimetric inequality has been done in Euclidean spaces, it is possible to extend this inequality to Minkowski spaces. In that case, one needs to define the notion of surface area and volume in Minkowski spaces.

A Minkowski space is a finite-dimensional linear normed space. Thus, this space can be considered as $\mathbb{R}^n$ equipped with arbitrary norm. Therefore, any origin-symmetric convex body can be taken as a *unit sphere* in order to define a Minkowski space.

Minkowski geometry is the study of the geometry of Minkowski spaces. Minkowski geometry is a non-Euclidean geometry which differs from our usual Euclidean view of space by being non-isotropic. This means that the space looks different in different directions. It explains why crystals grow in the way they do rather than spherically like soap bubbles. Not only does it have applications to crystal structures but many branches of physics where the forces are not uniform in all directions. Since in Minkowski geometry distance is not "uniform" in all directions, the Pythagoras' Theorem is no longer valid; however, the parallel axiom is still valid. Minkowski geometry is related to other areas of mathematics such as normed spaces, local theory of Banach spaces, convex and discrete geometry, integral geometry, Finsler geometry, Fourier analysis, and symplectic geometry. This field was also enriched from applied fields such as operations research, optimization, and computational geometry. The theory of Minkowski geometry has also applications to material science and crystal growth as described above.

Let λ be the *Lebesgue measure* induced by the standard Euclidean structure in $\mathbb{R}^d$. This measure is refereed as volume (area in $\mathbb{R}^2$). Then a Minkowski space possesses a *Haar measure* μ, and this measure (volume) is unique up to multiplication of the Lebesgue measure with a positive constant, i.e.,

$$\mu = \sigma_B \lambda.$$

Choosing the 'correct' multiple σ_B, which can depend on orientation, is not as easy as it seems at first glance. Also these two measures (volumes) μ and λ must coincide in the standard Euclidean space. There are two well-known notion of measures (*Holmes-Thompson* and *Busemann* measures) in Minkowski spaces.

3. Project Description

In recent years, Professor Horst Martini of Chemnitz Uniersity of Technology, Germany, and the proposer have been investigating properties of convex bodies in Euclidean/Minkowski spaces. They have already published several papers in a variety of refereed mathematical journals in this area.

Professor Horst Martini and the proposer will be investigating the possibility of extending the isoperimetric inequality to three or higher dimensional Minkowski spaces when the measure (volume) is defined in the Holmes-Thompson sense . During this visit Professor Horst Martini and the proposer are also planning to write a survey article on "Isoperimetri Inequalities in Minkowski Spaces."

Due to the nature and complexity of the proposed research project, face to face interaction between the investigators, Professor Horst Martini and the proposer, is essential. Personal communication will provide the opportunity to enhance the investigation process, as well as expedite the potential for publication of the results.

The proposer will also keep working on research/investigation projects with students on the topic of isoperimetric inequalities in Euclidean/Minkowski spaces. The proposer will supervise research/investigation projects of students in various degree programs in which such a program is a requirement. It is expected that students who do such projects will be interested and excited to continue learning the field and choose to undertake the further research.

Budget Summary and Justification

Offsite Research: The proposer will be visiting Chemnitz University of Technology, Germany for two weeks in the summer of 2009 to work on the research project (described in the proposal) with Professor Horst Martini. All efforts will be made to keep costs to a minimum. Rates are estimated based on previous years expenses. Receipts of expenses will be provided for reimbursement.

Air ticket (round trip): Houston-Frankfurt-Dresden $1,100.00

Train (round trip): Dresden-Chemnitz $60.00

Lodgings: $70.00 per day: Total of $1,080.00

Meals; $60.00 per day: Total of $840.00

Miscellaneous (parking, visa application fee, etc); $300.00

Amount requested for offsite research: $3,380.00

Total amount requested for the proposal: $3,380.00