from pp. 347-348 Cheney & Kincaid

7.5 Adams-Bashforth-Moulton Methods

A Predictor-Corrector Scheme

The procedures explained so far have solved the initial-value problem

$$\begin{cases} X' = F(X) \\ X(a) = S \quad \text{(given)} \end{cases} \tag{1}$$

by means of **single-step** numerical methods. In other words, if the solution $X(t)$ is known at a particular point t, then $X(t+h)$ can be computed with no knowledge of the solution at points earlier than t. The Runge-Kutta and Taylor series methods compute $X(t+h)$ in terms of $X(t)$ and various values of F.

Single-Step versus Multi-Step Methods

More efficient methods can be devised if several values $X(t), X(t-h), X(t-2h), \ldots$ are used in computing $X(t+h)$. Such methods are called **multi-step** methods. They have the obvious drawback that at the beginning of the numerical solution, no prior values of X are available. So it is usual to start a numerical solution with a single-step method, such as the Runge-Kutta procedure, and transfer to a multistep procedure for efficiency as soon as enough starting values have been computed.

An example of a multistep formula is known as the **Adams-Bashforth formula** (see Section 7.3 (p. 325) and the related Computer Exercises 7.3.2–4). It is

Adams-Bashforth Predictor Step

$$\widetilde{X}(t+h) = X(t) + \frac{h}{24}\{55F[X(t)] - 59F[X(t-h)] + 37F[X(t-2h)] - 9F[X(t-3h)]\} \tag{2}$$

Here, $\widetilde{X}(t+h)$ is the predicted value of $X(t+h)$ computed by using Formula (2). If the solution X has been computed at the four points t, $t-h$, $t-2h$, and $t-3h$, then Formula (2) can be used to compute $\widetilde{X}(t+h)$. If this is done systematically, then only *one* evaluation of F is required for each step. This represents a considerable savings over the fourth-order Runge-Kutta procedure; the latter requires *four* evaluations of F per step. (Of course, a consideration of truncation error and stability might permit a larger step size in the Runge-Kutta method and make it much more competitive.)

In practice, Formula (2) is never used by itself. Instead, it is used as a *predictor*, and then another formula is used as a *corrector*. The corrector that is usually used with Formula (2) is the **Adams-Moulton formula**:

Adams-Moulton Corrector Step

$$X(t+h) = X(t) + \frac{h}{24}\{9F[\widetilde{X}(t+h)] + 19F[X(t)] - 5F[X(t-h)] + F[X(t-2h)]\} \tag{3}$$

Thus, Formula (2) predicts a tentative value of $X(t+h)$, and Formula (3) computes this X value more accurately. The combination of the two formulas results in a **predictor-corrector scheme.**

With initial values of X specified at a, three steps of a Runge-Kutta method can be performed to determine enough X values that the Adams-Bashforth-Moulton procedure can begin. The fourth-order Adams-Bashforth and Adams-Moulton formulas, started with the fourth-order Runge-Kutta method, are referred to as the **Adams-Moulton method.** Predictor and corrector formulas of the same order are used so that only one application of the corrector formula is needed. Some suggest iterating the corrector formula, but experience has demonstrated that the best overall approach is only *one* application per step.

Adams-Moulton (Predictor-Corrector) Method