Obtaining Numerical Quadrature Formulae (n-points)

Let $f(x) = f(0) + x\,f'(0) + x^2\,f''(0)/2! + x^3\,f'''(0)/3! + \ldots$ be a given function for which

we want to obtain the integral:

$$I = \int f(x)\,dx \quad \text{using} \quad I \sim \sum w_i\,f(x_i)$$

where $\{x_i\}$ are the nodes and $\{w_i\}$ are the weights for the formula. Now, since

$$\int f(x)\,dx = f(0)\int dx + f'(0)\int x\,dx + f''(0)/2!\int x^2\,dx + f'''(0)/3!\int x^3\,dx + \ldots \text{ and}$$

$$\int f(x)\,dx = 2\,f(0) + 0\,f'(0) + 2/3\,f''(0)/2! + 0\,f'''(0)/3! + 2/5\,f^{iv}(0)/4! + \ldots$$

Now, $f(x_i) = f(0) + x_i\,f'(0) + x_i^2\,f''(0)/2! + x_i^3\,f'''(0)/3! + \ldots$ and hence

$$\sum w_i\,f(x_i) = f(0)\sum w_i + f'(0)\sum w_i\,x_i + f''(0)\sum w_i\,x_i^2/2! + f'''(0)\sum w_i\,x_i^3/3! + \ldots$$

So, if we choose :

$$\sum w_i = 2$$

$$\sum w_i\,x_i = 0$$

$$\sum w_i\,x_i^2 = 2/3$$

$$\sum w_i\,x_i^3 = 0$$

$$\sum w_i\,x_i^4 = 2/5$$

$$\vdots$$

or in general, we could say:

$$\sum w_i\,x_i^p = \begin{cases} 2/(p+1) & \text{for p even} \\ 0 & \text{for p odd} \end{cases}$$

and we can obtain a formula by solving these equations for the unknown nodes $\{x_i\}$ and unknown weights $\{w_i\}$. If the p-th equation is satisfied (and all of those equations preceding it) but not the p+1st equation, we say that the formula *is of degree p*. If original integral is over the range [a,b] instead of [-1,1] , then let original problem be $\int_a^b f(z)\,dz$ for example and use the linear transformation: $z = (b+a)/2 + (b-a)\,x/2$. Then the formula becomes: $I \sim (b-a)/2\ \sum w_i\,f(z_i)$ where $z_i = (b+a)/2 + (b-a)\,x_i/2$.

RESULTS, COMPARISONS, AND CONCLUSIONS

The two methods presented in Tables I and II have been applied to a few simple quadrature problems for which the true value of the integral is known. The results are compared with those obtained using the following well-known methods: Romberg's method, the Clenshaw-Curtis method, Patterson's method, and a method composed of Gauss formulae. The composite Gauss method consists of Gauss formulae of degrees one, three, seven, and fifteen. The total number of function evaluations required at each stage are therefore one, three, nine, and twenty-three respectively.

The following set of problems are considered:

$$\int_{-1}^{1} \frac{8\,dx}{x^2 + 2x + 5} = \pi, \tag{1}$$

$$\int_{0}^{1} x^{1/2}\,dx = 2/3, \tag{2}$$

$$\int_{0}^{1} x^{3/2}\,dx = 2/5, \tag{3}$$

$$\int_{1}^{2} \frac{dx}{x} = \ln 2, \tag{4}$$

and

$$\int_{-1}^{1} \frac{dx}{1 + x^2} = \pi/2. \tag{5}$$

Table III

Errors Obtained Using Certain Quadrature Methods

Problem No.	n	Romberg's Method	n	Clenshaw-Curtis Method	n	Method I from Table I	n	Patterson's Method	n	Composite Gauss Method (1-3-7-15)	n	Method II from Table II
1	3	8.26×10^{-3}	3	8.26×10^{-3}	3	8.26×10^{-3}	1	5.84×10^{-2}	1	5.84×10^{-2}	2	5.94×10^{-3}
	5	5.25×10^{-4}	5	2.36×10^{-4}	5	2.40×10^{-5}	3	5.25×10^{-4}	3	5.25×10^{-4}	5	6.04×10^{-6}
	9	6.87×10^{-6}	9	3.31×10^{-9}	9	6.91×10^{-10}	7	1.84×10^{-8}	9	2.66×10^{-9}	11	6.32×10^{-13}
	17	1.17×10^{-8}	17	–	17	–	15	–	23	–	23	–
2	3	2.86×10^{-2}	3	2.86×10^{-2}	3	2.86×10^{-2}	1	4.04×10^{-2}	1	4.04×10^{-2}	2	7.22×10^{-3}
	5	8.91×10^{-3}	5	2.07×10^{-3}	5	4.53×10^{-3}	3	2.51×10^{-3}	3	2.51×10^{-3}	5	3.41×10^{-4}
	9	3.06×10^{-3}	9	2.24×10^{-4}	9	7.61×10^{-4}	7	1.42×10^{-4}	9	2.46×10^{-4}	11	2.48×10^{-5}
	17	1.07×10^{-3}	17	2.69×10^{-5}	17	1.27×10^{-4}	15	7.05×10^{-6}	23	2.77×10^{-5}	23	2.13×10^{-6}
3	3	2.37×10^{-3}	3	2.37×10^{-3}	3	2.37×10^{-3}	1	4.64×10^{-2}	1	4.64×10^{-2}	2	1.22×10^{-3}
	5	3.03×10^{-4}	5	1.25×10^{-5}	5	9.21×10^{-5}	3	1.88×10^{-4}	3	1.88×10^{-4}	5	2.76×10^{-6}
	9	4.96×10^{-5}	9	7.79×10^{-7}	9	4.53×10^{-6}	7	1.03×10^{-6}	9	3.60×10^{-6}	11	7.71×10^{-8}
	17	8.62×10^{-6}	17	2.73×10^{-8}	17	2.27×10^{-7}	15	1.78×10^{-9}	23	9.29×10^{-8}	23	1.63×10^{-9}
4	3	1.30×10^{-3}	3	1.30×10^{-3}	3	1.30×10^{-3}	1	2.65×10^{-2}	1	2.65×10^{-2}	2	8.39×10^{-4}
	5	2.74×10^{-5}	5	9.93×10^{-6}	5	9.68×10^{-7}	3	2.55×10^{-5}	3	2.55×10^{-5}	5	2.18×10^{-7}
	9	2.97×10^{-7}	9	6.40×10^{-10}	9	6.44×10^{-12}	7	2.39×10^{-10}	9	1.99×10^{-11}	11	–
	17	1.36×10^{-9}	17	–	17	–	15	–	23	–	23	–
5	3	9.59×10^{-2}	3	9.59×10^{-2}	3	9.59×10^{-2}	1	4.29×10^{-1}	1	4.29×10^{-1}	2	7.08×10^{-2}
	5	1.08×10^{-2}	5	6.98×10^{-3}	5	2.54×10^{-3}	3	1.25×10^{-2}	3	1.25×10^{-2}	5	9.99×10^{-4}
	9	4.38×10^{-4}	9	9.77×10^{-6}	9	1.06×10^{-6}	7	3.35×10^{-5}	9	1.11×10^{-5}	11	1.37×10^{-7}
	17	5.17×10^{-6}	17	5.30×10^{-10}	17	4.88×10^{-11}	15	1.87×10^{-10}	23	8.46×10^{-12}	23	–

– Indicates an error less than 1×10^{-13} n = Total number of function evaluations.

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