The Solution of the 4th–Order Runge-Kutta Equations of Condition

In order to determine a 4th–Order Runge-Kutta with only four evaluations per step (a.k.a., the number of stages), you would have to solve eight nonlinear algebraic equations in ten unknowns. A simplified solution is shown below.

Any 4th-order RK method with four stages must normally satisfy the following eight equations.

\[
\begin{align*}
\alpha_3 &= 1 \\
c_0 &= \frac{1 - 2\alpha_2 + \alpha_1(\alpha_1 - \alpha_2)}{12\alpha_1 \alpha_2} \\
c_1 &= \frac{-1 + 2\alpha_2}{12(-1 + \alpha_1)(\alpha_1 - \alpha_2)} \\
c_2 &= \frac{1 - 2\alpha_1}{12(\alpha_1 - \alpha_2)(-1 + \alpha_2)\alpha_2} \\
c_3 &= \frac{3 - 4\alpha_2 + \alpha_1(\alpha_1 - \alpha_2)}{12(-1 + \alpha_1)(-1 + \alpha_2)} \\
\beta_{2,1} &= \frac{(\alpha_1 - \alpha_2)\alpha_2}{2\alpha_1(-1 + 2\alpha_1)} \\
\beta_{3,1} &= \frac{(-1 + \alpha_1)(\alpha_1 - 2\alpha_1 + 5\alpha_2 - 4\alpha_2^2)}{2\alpha_1(\alpha_1 - \alpha_2)(3 - 4\alpha_2 + \alpha_1(\alpha_1 - \alpha_2))} \\
\beta_{3,2} &= \frac{(-1 + \alpha_1)(\alpha_1 - 2\alpha_1)(-1 + \alpha_2)}{(\alpha_1 - \alpha_2)\alpha_2(3 - 4\alpha_2 + \alpha_1(\alpha_1 - \alpha_2))}
\end{align*}
\]

In order to obtain your method, select values for \(\alpha_1\) and \(\alpha_2\), then calculate the remaining coefficients as shown above. Note that you cannot normally choose any of the following:

\(\alpha_1 = 0\), \(\alpha_1 = 1\), \(\alpha_1 = \frac{1}{2}\), \(\alpha_1 = \alpha_2\), \(\alpha_2 = 0\), \(\alpha_2 = 1\), or \(\alpha_2 = \frac{3 - 4\alpha_1}{4 - 6\alpha_1}\), because that would cause some of the above denominators to be zero.

Note, there is a special solution for the equations of condition when \(\alpha_1 = \alpha_2\), but then you would have to satisfy the following conditions, in order to find a 4th-order method.

\[
\begin{align*}
c_0 &= \frac{1}{6}, \quad c_1 = \frac{1}{3}(2 - 3c_2), \quad c_3 = \frac{1}{6}, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{1}{2}, \quad \alpha_3 = 1, \quad \beta_{2,1} = \frac{1}{6c_2}, \quad \beta_{3,1} = 1 - 3c_2, \quad \beta_{3,2} = 3c_2
\end{align*}
\]

The “classical” 4th-order RK method has \(\alpha_1 = \frac{1}{2}\) and \(\alpha_2 = \frac{1}{2}\), but I do not want you to use that method (or any method with \(\alpha_1 = \frac{1}{2}\) and \(\alpha_2 = \frac{1}{2}\)). Please see the instructions for Assignment #4.