The Solution of the 4th–Order Runge-Kutta Equations of Condition

In order to determine a 4th–Order Runge-Kutta with only four **evaluations per step** (a.k.a., the number of stages), you would have to solve eight nonlinear algebraic equations in ten unknowns. A simplified solution is shown below.

Any 4th-order RK method with four stages must *normally* satisfy the following eight equations.

$$\alpha_3 = 1$$

$$c_0 = \frac{1 - 2\alpha_2 + \alpha_1(-2 + 6\alpha_2)}{12\alpha_1\alpha_2}$$

$$c_1 = \frac{-1 + 2\alpha_2}{12(-1 + \alpha_1)\alpha_1(\alpha_1 - \alpha_2)}$$

$$c_2 = \frac{1 - 2\alpha_1}{12(\alpha_1 - \alpha_2)(-1 + \alpha_2)\alpha_2}$$

$$c_3 = \frac{3 - 4\alpha_2 + \alpha_1(-4 + 6\alpha_2)}{12(-1 + \alpha_1)(-1 + \alpha_2)}$$

$$\beta_{2,1} = \frac{(\alpha_1 - \alpha_2)\alpha_2}{2\alpha_1(-1 + 2\alpha_1)}$$

$$\beta_{3,1} = \frac{(-1 + \alpha_1)(-2 + \alpha_1 + 5\alpha_2 - 4\alpha_2^2)}{2\alpha_1(\alpha_1 - \alpha_2)\big(3 - 4\alpha_2 + \alpha_1(-4 + 6\alpha_2)\big)}$$

$$\beta_{3,2} = \frac{(-1 + \alpha_1)(-1 + 2\alpha_1)(-1 + \alpha_2)}{(\alpha_1 - \alpha_2)\alpha_2(3 - 4\alpha_2 + \alpha_1(-4 + 6\alpha_2))}$$

In order to obtain your method, select values for α_1 and α_2, then calculate the remaining coefficients as shown above. Note that you cannot normally choose any of the following:

$\alpha_1 = 0,\ \alpha_1 = 1,\ \alpha_1 = \frac{1}{2},\ \alpha_1 = \alpha_2,\ \alpha_2 = 0,\ \alpha_2 = 1,\ or\ \alpha_2 = \frac{3-4\alpha_1}{4-6\alpha_1}$, because that would cause some of the above denominators to be zero.

Note, there is a special solution for the equations of condition when $\alpha_1 = \alpha_2$, but then you would have to satisfy the following conditions, in order to find a 4th-order method.

$$c_0 = \frac{1}{6}, c_1 = \frac{1}{3}(2 - 3c_2), c_3 = \frac{1}{6}, \alpha_1 = \frac{1}{2}, \alpha_2 = \frac{1}{2}, \alpha_3 = 1, \beta_{2,1} = \frac{1}{6c_2}, \beta_{3,1} = 1 - 3c_2, \beta_{3,2} = 3c_2$$

The "classical" 4th-order RK method has $\alpha_1 = \frac{1}{2}\ and\ \alpha_2 = \frac{1}{2}$. but I do not want you to use that method (or any method with $\alpha_1 = \frac{1}{2}\ and\ \alpha_2 = \frac{1}{2}$). Please see the instructions for Assignment #4.