Supplementary Exercises and Exam Questions
Discrete Mathematics with Applications, 3rd Edition
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Chapter 8

1. In a *Double Tower of Hanoi with Adjacency Requirement* there are three poles in a row and $2n$ disks, two of each of n different sizes, where n is any positive integer. Initially pole A (at one end of the row) contains all the disks, placed on top of each other in *pairs* of decreasing size. Disks may only be transferred one-by-one from one pole to an *adjacent* pole and at no time may a larger disk be placed on top of a smaller one. However a disk may be placed on top of another one of the same size. Let C be the pole at the other end of the row and let

$$s_n = \begin{bmatrix} \text{the minimum number of moves} \\ \text{needed to transfer a tower of } 2n \\ \text{disks from pole } A \text{ to pole } C \end{bmatrix}.$$

   (a) Find s_1 and s_2.

   (b) Find a recurrence relation expressing s_k in terms of s_{k-1} for all integers $k \geq 2$. Justify your answer carefully.

2. In a *Triple Tower of Hanoi*, there are three poles in a row and $3n$ disks, *three* of each of n different sizes, where n is any positive integer. Initially, one of the poles contains all the disks placed on top of each other in triples of decreasing size. Disks are transferred one by one from one pole to another, but at no time may a larger disk be placed on top of a smaller disk. However, a disk may be placed on top of one of the same size. Let t_n be the minimum number of moves needed to transfer a tower of $3n$ disks from one pole to another. Find a recurrence relation for $t_1, t_2, t_3, \ldots$. Justify your answer carefully.

3. A single pair of rabbits (male and female) is born at the beginning of a year. Assume the following conditions: (a) Rabbit pairs are not fertile during their first *two* months of life, but thereafter they give birth to *four* new male/female pairs at the end of every month; (b) No deaths occur. Let $s_n =$ the number of pairs of rabbits alive at the end of month n, for each integer $n \geq 1$, and let $s_0 = 1$. Find a recurrence relation for $s_0, s_1, s_{2,\ldots}$.Justify your answer carefully.

4. Suppose a certain amount of money is deposited into an account paying 4% annual interest, compounded quarterly. For each positive integer n, let $S_n =$ the amount on deposit at the end of the nth quarter, and let S_0 be the initial amount deposited.

   (a) Find a recurrence relation for $S_0, S_1, S_2, \ldots$, assuming no additional deposits or withdrawals for a 4-year period.

   (b) If $S_0 = \$5000$, find the amount of money on deposit at the end of 4 years.

   (c) Find the APR for the account.

5. Consider the set S of all strings of a's and b's. For each integer $n \geq 0$, let

$$\boxed{a_n = \text{the number of strings of length n that do not contain the pattern } bb.}$$

   Find a recurrence relation for $a_1, a_2, a_3, \ldots$. Explain your answer carefully.

6. A sequence $a_1, a_2, a_3, \ldots$ is defined as follows:

$$a_1 = 3, \qquad \text{and} \qquad a_k = 4a_{k-1} + 2 \quad \text{for all integers } k \geq 2.$$

(a) Find a_1, a_2, and a_3.

(b) Supposing that $a_5 = 4^4 \cdot 3 + 4^3 \cdot 2 + 4^2 \cdot 2 + 4 \cdot 2 + 2$, find a similar numerical expression for a_6 by substituting the right-hand side of this equation in place of a_5 in the equation

$$a_6 = 4 \cdot a_5 + 2.$$

(c) Guess an explicit formula for a_n.

7. A sequence $c_0, c_1, c_2, \ldots$ is defined as follows:

$$c_0 = 1 \qquad \text{and} \qquad c_k = 7c_{k-1} + 2 \quad \text{for each integer } k \geq 1.$$

(a) Find c_1 and c_2.

(b) Use one of the reference formulas given at the end of this exam to simplify the expression

$$7^n + 2 \cdot 7^{n-1} + \cdots + 2 \cdot 7^2 + 2 \cdot 7 + 2.$$

(c) Use iteration to guess an explicit formula for the sequence $c_0, c_1, c_2, \ldots$.

8. Use iteration to find an explicit formula for the sequence $b_0, b_1, b_2, \ldots$ defined recursively as follows:

$$\begin{aligned} b_k &= 2b_{k-1} + 3 \quad \text{for all integers } k \geq 1 \\ b_0 &= 1. \end{aligned}$$

If appropriate, simplify your answer using one of the following reference formulas:

(a) $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for all integers $n \geq 1$.

(b) $1 + r + r^2 + \cdots + r^m = \dfrac{r^{m+1} - 1}{r - 1}$ for all integers $m \geq 0$ and all real numbers $r \neq 1$.

9. A sequence is defined recursively as follows:

$$a_0 = 2 \qquad \text{and} \qquad a_k = 4a_{k-1} + 1 \quad \text{for all } k \geq 1.$$

It is proposed that an explicit formula for this sequence is

$$a_n = \frac{7 \cdot 4^n - 1}{3}.$$

Use mathematical induction to check whether this proposed formula is correct.

10. A sequence is defined recursively as follows:

$$\begin{aligned} s_k &= 5s_{k-1} + 1 \quad \text{for all integers } k \geq 1 \\ s_0 &= 1. \end{aligned}$$

Use mathematical induction to verify that this sequence satisfies the explicit formula

$$s_n = \frac{5^{n+1} - 1}{4} \qquad \text{for all integers } n \geq 0.$$

11. A sequence $a_0, a_1, a_2, \ldots$ satisfies the recurrence relation $a_k = 4a_{k-1} - 3a_{k-2}$ with initial conditions $a_0 = 1$ and $a_1 = 2$. Find an explicit formula for the sequence.

12. A sequence $b_1, b_2, b_3, \ldots$ satisfies the recurrence relation $b_k = 2b_{k-1} + 8b_{k-2}$ with initial conditions $b_1 = 1$ and $b_2 = 0$. Find an explicit formula for the sequence.

13. A sequence $c_0, c_1, c_2, \ldots$ satisfies the recurrence relation $c_k = 6c_{k-1} - 9c_{k-2}$ with initial conditions $c_0 = 1$ and $c_1 = 6$. Find an explicit formula for the sequence.

14. A sequence $d_1, d_2, d_3, \ldots$ satisfies the recurrence relation $d_k = 8d_{k-1} - 16d_{k-2}$ with initial conditions $d_1 = 0$ and $d_2 = 1$. Find an explicit formula for the sequence.

15. Define a set S recursively as follows:

    I. BASIS: $11 \in S$

    II. RECURSION:

    a. If $s \in S$, then $0s \in S$ and $s0 \in S$

    b. If x is any string (including the null string) such that $1x1 \in S$, then $10x1 \in S$ and $1x01 \in S$

    III. RESTRICTION: No strings other than those derived from I and II are in S.

    a. Is $00100010 \in S$? b. Is $011011 \in S$?

16. Define a set S recursively as follows:

    I. BASIS: $\epsilon \in S$

    II. RECURSION: If s and t are in S, then

    a. $0s \in S$ b. $s0 \in S$ c. $1s1t \in S$ d. $s1t1 \in S$

    III. RESTRICTION: No strings other than those derived from I and II are in S.

    Use structural induction to prove that every string in S contains an even number of 1's.

17. Use the recursive definition of summation together with mathematical induction to prove that for all positive integers n, if $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are real numbers, then

$$\sum_{k=1}^{n}(2a_k - 3b_k) = 2\sum_{k=1}^{n} a_k - 3\sum_{k=1}^{n} b_k.$$